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The Electromagnetic Energy Momentum Tensor and its Uniqueness

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Abstract

The uniqueness of the electromagnetic energy momentum tensor is established under general conditions.

1. *Introduction*

In electromagnetic field theory and in general relativity the role played by the electromagnetic energy momentum tensor†

$$T^{ij} = -F^{ih}F^{j}{}_{h} + \tfrac{1}{4}g^{ij}(F^{ab}F_{ab}) \tag{1.1}$$

where‡

$$F_{ij} = \psi_{i,j} - \psi_{j,i} \tag{1.2}$$

and ψ_i is an arbitrary vector field, is well known. Some of the more important properties which T^{ij} enjoys are

(a) T^{ij} is symmetric, i.e.

$$T^{ij} = T^{ji}; \tag{1.3}$$

(b) whenever the source-free Maxwell equations§

$$F^{ij}{}_{|j} = 0 \tag{1.4}$$

† Latin indices run from 1 to n. g_{ij} is the metric of an n-dimensional Riemannian space. The summation convention is used throughout.

‡ A comma denotes partial differentiation.

§ A vertical bar denotes covariant differentiation. The second set of Maxwell's equations $F_{ij|k} + F_{ki|j} + F_{jk|i} = 0$ are identically satisfied by virtue of (1.2).

are satisfied, the divergence of T^{ij}, i.e. $T^{ij}{}_{|j}$, vanishes by virtue of the identity

$$T^{ij}{}_{|j} = F^{ih} F_h{}^j{}_{|j}\,; \tag{1.5}$$

(c) T^{ij} is trace-free in four dimensions, i.e.

$$g_{ij} T^{ij} = 0 \qquad \text{for } n = 4. \tag{1.6}$$

Guided by the properties (1.3) and (1.5) the present work is devoted to establishing that in n-dimensions T^{ij} given by (1.1) is essentially the unique solution to the following problem. To find all tensors B^{ij} for which

(i) B^{ij} is a concomitant of g_{ab}, ψ_a and $\psi_{a,b}$, i.e.

$$B^{ij} = B^{ij}(g_{ab}; \psi_a; \psi_{a,b})\,; \tag{1.7}$$

(ii) B^{ij} is symmetric, i.e.

$$B^{ij} = B^{ji}\,; \tag{1.8}$$

(iii) $B^{ij}{}_{|j}$ vanishes whenever (1.4) is valid in the sense that†

$$B^{ij}{}_{|j} = \alpha^{ih} F_h{}^j{}_{|j}. \tag{1.9}$$

where α^{ih} is a tensor and a concomitant of g_{ab}, ψ_a and $\psi_{a,b}$, i.e.

$$\alpha^{ih} = \alpha^{ih}(g_{ab}; \psi_a; \psi_{a,b}). \tag{1.10}$$

Alternative conditions under which T^{ij} is determined uniquely have been discussed by Fock (1964, p. 411) and Collinson (1969).

2. *The Uniqueness of the Electromagnetic Energy Momentum Tensor*

In this section we shall find all tensors B^{ij} which satisfy (1.7), (1.8) and (1.9).

In view of the fact that B^{ij} and α^{ih} are tensor concomitants certain invariance identities must be satisfied (Rund, 1966), viz.

$$\frac{\partial B^{ij}}{\partial \psi_{r,s}} + \frac{\partial B^{ij}}{\partial \psi_{s,r}} = 0, \tag{2.1}$$

$$\delta_r^l B^{sm} + \delta_r^m B^{ls} + 2\frac{\partial B^{lm}}{\partial g_{sk}} g_{rk} + \frac{\partial B^{lm}}{\partial \psi_s}\psi_r + \frac{\partial B^{lm}}{\partial \psi_{s,k}} F_{rk} = 0, \tag{2.2}$$

$$\frac{\partial \alpha^{ij}}{\partial \psi_{r,s}} + \frac{\partial \alpha^{ij}}{\partial \psi_{s,r}} = 0, \tag{2.3}$$

and

$$\delta_r^l \alpha^{sm} + \delta_r^m \alpha^{ls} + 2\frac{\partial \alpha^{lm}}{\partial g_{sk}} g_{rk} + \frac{\partial \alpha^{lm}}{\partial \psi_s}\psi_r + \frac{\partial \alpha^{lm}}{\partial \psi_{s,k}} F_{rk} = 0, \tag{2.4}$$

† Clearly (1.7) guarantees that $B^{ij}{}_{|j}$ is at worst linear in $F_{ij,k}$.

Written out in detail, (1.9) reads

$$\frac{\partial B^{ij}}{\partial \psi_{a,b}} \psi_{a,bj} + \frac{\partial B^{ij}}{\partial \psi_a} \psi_{a,j} + \frac{\partial B^{ij}}{\partial g_{ab}} g_{ab,j} + \Gamma^i_{aj} B^{aj} + \Gamma^j_{aj} B^{ia}$$

$$= \alpha^{ih} g^{jk} [\psi_{h,kj} - \psi_{k,hj} - \Gamma^a_{hj} F_{ak} - \Gamma^a_{kj} F_{ha}]. \quad (2.5)$$

Differentiation of (2.5) with respect to $\psi_{r,st}$ yields

$$\frac{1}{2}\left(\frac{\partial B^{it}}{\partial \psi_{r,s}} + \frac{\partial B^{is}}{\partial \psi_{r,t}}\right) = \alpha^{ir} g^{st} - \tfrac{1}{2}\alpha^{is} g^{rt} - \tfrac{1}{2}\alpha^{it} g^{rs}, \quad (2.6)$$

while differentiation of (2.5) with respect to $g_{rs,t}$ gives

$$\frac{\partial B^{it}}{\partial g_{rs}} + \tfrac{1}{2} g^{is} B^{rt} + \tfrac{1}{2} g^{ir} B^{st} - \tfrac{1}{2} g^{it} B^{rs} + \tfrac{1}{2} g^{rs} B^{it}$$

$$= \tfrac{1}{2}\alpha^{ir} F^{ts} + \tfrac{1}{2}\alpha^{is} F^{tr} + \tfrac{1}{2}\alpha^{ik} F^{s}{}_{k} g^{rt} + \tfrac{1}{2}\alpha^{ik} F^{r}{}_{k} g^{st} - \tfrac{1}{2}\alpha^{ik} F^{t}{}_{k} g^{rs}. \quad (2.7)$$

If (2.6) and (2.7) are substituted in (2.5) we see that

$$\frac{\partial B^{ij}}{\partial \psi_a} \psi_{a,j} = 0. \quad (2.8)$$

Consequently (2.6), (2.7) and (2.8) are equivalent to (2.5).

Differentiation of (2.8) with respect to $\psi_{r,s}$ yields

$$\frac{\partial^2 B^{ij}}{\partial \psi_{r,s}\, \partial \psi_a} \psi_{a,j} + \frac{\partial B^{is}}{\partial \psi_r} = 0. \quad (2.9)$$

By virtue of (2.1), (2.9) implies that

$$\frac{\partial B^{is}}{\partial \psi_r} + \frac{\partial B^{ir}}{\partial \psi_s} = 0,$$

from which, in view of (1.8), it is easily established that

$$\frac{\partial B^{is}}{\partial \psi_r} = 0$$

i.e.

$$B^{is} = B^{is}(g_{ab};\, \psi_{a,b}). \quad (2.10)$$

We now turn to an analysis of (2.6). In (2.6) we interchange i with s, and also i with t to find

$$\frac{1}{2}\left(\frac{\partial B^{st}}{\partial \psi_{r,i}} + \frac{\partial B^{is}}{\partial \psi_{r,t}}\right) = \alpha^{sr} g^{it} - \tfrac{1}{2}\alpha^{si} g^{rt} - \tfrac{1}{2}\alpha^{st} g^{ri}, \quad (2.11)$$

and

$$\frac{1}{2}\left(\frac{\partial B^{it}}{\partial \psi_{r,s}}+\frac{\partial B^{ts}}{\partial \psi_{r,i}}\right)=\alpha^{tr}g^{si}-\tfrac{1}{2}\alpha^{ts}g^{ri}-\tfrac{1}{2}\alpha^{ti}g^{rs}, \tag{2.12}$$

respectively. Adding (2.6) and (2.11) and subtracting (2.12), we see that

$$\frac{\partial B^{is}}{\partial \psi_{r,t}}=\alpha^{ir}g^{st}-D^{is}g^{rt}+\tfrac{1}{2}(\alpha^{ti}-\alpha^{it})g^{rs}+\alpha^{sr}g^{it}$$
$$+\tfrac{1}{2}(\alpha^{ts}-\alpha^{st})g^{ri}-\alpha^{tr}g^{si} \tag{2.13}$$

where

$$D^{is}=\tfrac{1}{2}(\alpha^{is}+\alpha^{si}).$$

In (2.13) we interchange r and t and make use of (2.1) to find

$$D^{ir}g^{st}+D^{it}g^{rs}+D^{rs}g^{it}+D^{st}g^{ri}-2D^{is}g^{rt}-2D^{rt}g^{is}=0. \tag{2.14}$$

Multiplication of (2.14) by g_{ir} yields

$$(n-2)D^{st}+g^{st}(g_{ir}D^{ir})=0, \tag{2.15}$$

from which, for $n\neq 2$, it follows that

$$D^{st}=0$$

i.e.

$$\alpha^{st}=-\alpha^{ts}. \tag{2.16}$$

Restricting our considerations to the case $n\geqslant 3$, we see that, in view of (2.16), (2.13) reduces to

$$\frac{\partial B^{is}}{\partial \psi_{r,t}}=\alpha^{ir}g^{st}+\alpha^{ti}g^{rs}+\alpha^{sr}g^{it}+\alpha^{ts}g^{ri}+\alpha^{rt}g^{si}. \tag{2.17}$$

If we multiply (2.17) by g_{ri} we see that

$$\frac{\partial B_i^{\ s}}{\partial \psi_{i,t}}=(n-1)\alpha^{ts},$$

which, by virtue of (2.10), implies that

$$\alpha^{ts}=\alpha^{ts}(g_{ab};\psi_{a,b}). \tag{2.18}$$

We now differentiate (2.17) with respect to $\psi_{a,b}$ and find

$$\frac{\partial^2 B^{is}}{\partial \psi_{a,b}\,\partial \psi_{r,t}}=\alpha^{ir;ab}g^{st}+\alpha^{ti;ab}g^{rs}+\alpha^{sr;ab}g^{it}+\alpha^{ts;ab}g^{ri}+\alpha^{rt;ab}g^{si} \tag{2.19}$$

where

$$\alpha^{ir;ab}=\partial\alpha^{ir}/\partial\psi_{a,b}. \tag{2.20}$$

From (2.3) and (2.16) we clearly have

$$\alpha^{ir;ab} = -\alpha^{ri;ab} = -\alpha^{ir;ba}.$$

The equation obtained from (2.19) by interchanging a with r and b with t is subtracted from (2.19) to give

$$\alpha^{ir;ab}g^{st} + \alpha^{ti;ab}g^{rs} + \alpha^{sr;ab}g^{it} + \alpha^{ts;ab}g^{ri} + \alpha^{rt;ab}g^{si} = \alpha^{ia;rt}g^{sb} + \alpha^{bi;rt}g^{as} + \alpha^{sa;rt}g^{ib} + \alpha^{bs;rt}g^{ai} + \alpha^{ab;rt}g^{si}. \quad (2.21)$$

We multiply (2.21) by g_{st} and let

$$\mu^{br} = \alpha^{bs;rt}g_{st}$$

to find

$$(n-1)\alpha^{ir;ab} + \alpha^{ia;br} + \alpha^{bi;ar} + \alpha^{ab;ir} = \mu^{br}g^{ia} - \mu^{ar}g^{ib}. \quad (2.22)$$

Multiplication of (2.22) by g_{ib} thus yields

$$\mu^{ra} = \mu^{ar}. \quad (2.23)$$

Multiplication of (2.22) by g_{rb}, account being taken of (2.23), gives rise to

$$\mu^{ar} = \lambda g^{ar} \quad (2.24)$$

where

$$\lambda = \frac{1}{n}(g_{ij}\mu^{ij}) = \lambda(g_{ab}; \psi_{a,h}), \quad (2.25)$$

in view of (2.18).

When (2.24) is substituted in (2.22) we find

$$(n-1)\alpha^{ir;ab} + \alpha^{ia;br} + \alpha^{bi;ar} + \alpha^{ab;ir} = \lambda(g^{br}g^{ia} - g^{ar}g^{ib}). \quad (2.26)$$

In (2.26) we cycle on abi to find

$$\tfrac{1}{3}(n-1)[\alpha^{ir;ab} + \alpha^{ar;bi} + \alpha^{br;ia}] + [\alpha^{ia;br} + \alpha^{bi;ar} + \alpha^{ab;ir}] = 0. \quad (2.27)$$

We subtract (2.27) from (2.26) and see that

$$\tfrac{1}{3}(n-1)[2\alpha^{ir;ab} - \alpha^{ar;bi} - \alpha^{br;ia}] = \lambda(g^{br}g^{ia} - g^{ar}g^{ib}). \quad (2.28)$$

The equation obtained from (2.28) by interchanging a with i is added to (2.28) to yield

$$(n-1)(\alpha^{ir;ab} + \alpha^{ar;ib}) = \lambda(2g^{br}g^{ai} - g^{ar}g^{ib} - g^{ir}g^{ab}). \quad (2.29)$$

The equation obtained from (2.29) by interchanging i with r is subtracted from (2.29) to give

$$\tfrac{1}{3}(n-1)[2\alpha^{ir;ab} + \alpha^{ar;ib} + \alpha^{ai;br}] = \lambda(g^{br}g^{ai} - g^{ar}g^{bi}). \quad (2.30)$$

A comparison of (2.28) and (2.30) thus shows that

$$\alpha^{ai;br} = \alpha^{br;ai},$$

which, when substituted in (2.27), yields

$$(n+2)(\alpha^{ir;ab}+\alpha^{ar;bi}+\alpha^{br;ia})=0. \tag{2.31}$$

(2.31) is now substituted in (2.26) and gives rise to

$$\alpha^{ir;ab}=\frac{\lambda}{n-1}(g^{br}g^{ai}-g^{ar}g^{ib}). \tag{2.32}$$

We differentiate (2.32) with respect to $\psi_{c,d}$ and note (2.20) to find

$$\frac{\partial\lambda}{\partial\psi_{c,d}}(g^{br}g^{ai}-g^{ar}g^{ib})=\frac{\partial\lambda}{\partial\psi_{a,b}}(g^{dr}g^{ci}-g^{cr}g^{id}).$$

Multiplication of the latter by $g_{ai}g_{rb}$ yields

$$(n-2)(n+1)\frac{\partial\lambda}{\partial\psi_{c,d}}=0,$$

so that, by (2.25)

$$\lambda=\lambda(g_{ab})$$

in which case (Lovelock, 1969)

$$\lambda=(n-1)a \tag{2.33}$$

where a is a constant.

Substitution of (2.33) in (2.32) and integration yields

$$\alpha^{ir}=aF^{ir}+\beta^{ir} \tag{2.34}$$

where β^{ir} is an antisymmetric tensor and, by (2.18),

$$\beta^{ir}=\beta^{ir}(g_{ab}).$$

Thus, for $n>2$ (Lovelock, 1969)

$$\beta^{ir}=0,$$

in which case (2.34) reduces to

$$\alpha^{ir}=aF^{ir}. \tag{2.35}$$

We now multiply (2.7) by g_{rs} and use (2.35) to find

$$\frac{\partial B^{it}}{\partial g_{rs}}g_{rs}=\tfrac{1}{2}g^{it}B-\left(1+\frac{n}{2}\right)B^{it}+\frac{a}{2}(4-n)F^{ir}F^{t}{}_{r} \tag{2.36}$$

where

$$B=g_{it}B^{it}.$$

Also from (2.17) and (2.35) we have

$$\frac{\partial B^{it}}{\partial\psi_{r,s}}F_{rs}=a(g^{it}F^{ab}F_{ab}-4F^{ir}F^{t}{}_{r}). \tag{2.37}$$

However, from (2.2) and (2.10),

$$B^{it} = -\frac{\partial B^{it}}{\partial g_{rs}} g_{rs} - \frac{1}{2} \frac{\partial B^{it}}{\partial \psi_{r,s}} F_{rs},$$

which, by (2.36) and (2.37) reduces to

$$B^{it} = \frac{1}{n} g^{it} B + a \left[\frac{1}{n} g^{it} F^{ab} F_{ab} - F^{ir} F^{t}{}_{r} \right]. \tag{2.38}$$

However, from (2.17) and (2.35), we see that

$$\frac{\partial B}{\partial \psi_{r,t}} = a(4-n) F^{tr},$$

from which it can be shown that

$$B = -a \left(1 - \frac{n}{4}\right) F^{tr} F_{tr} + nb \tag{2.39}$$

where b is a constant. Substitution of (2.39) in (2.38) gives

$$B^{it} = g^{it} b + a [\tfrac{1}{4} g^{it} F^{cb} F_{cb} - F^{ir} F^{t}{}_{r}]. \tag{2.40}$$

We have thus proved the

Theorem: For $n > 2$ the only tensor which satisfies (1,7), (1.8) *and* (1.9) *is*

$$B^{it} = aT^{it} + bg^{it}$$

where a and b are constants.

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